Surface Renewal Based Formulation for Turbulent Boundary-Layer Flow

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A preliminary surface renewal based analysis of turbulent boundary-layer flow is presented that accounts for the effects of axial convection. This model provides a foundation for the unified analysis of both turbulent and laminar boundary-layer flow.

Introduction

WHEREAS momentum transfer for steady laminar boundary-layer flow can be readily handled by the solution of the appropriate momentum and continuity equation, a rigorous theoretical treatment of the turbulent boundary layer has not been set forth. The standard approach in analyzing this complex problem has involved the formulation of the time average momentum equation, i.e., the Reynolds equation, with information pertaining to the Reynolds stress, $\overline{u'v'}$, being supplied by phenomenological models or by the statistical theory of turbulence. Because the phenomenological models have not been based on a physically meaningful description of turbulent motion, these models do not provide insight into the actual mechanism. In connection with the statistical theory of turbulence, because of the complex nature of the turbulence mechanism, this approach has yet to provide a general model upon which to base an analysis.[1]

Recent flow visualization studies,[2-5] however, have provided a clearer picture of key characteristics of the mechanism of turbulent shear flow. Specifically, the turbulent flow within the important wall region has been described as consisting of bursting cycles which are distributed randomly in time and space. These bursts consist of 1) the inrush of high axial momentum fluid into the wall region, 2) an apparent period of unsteady molecular transfer, and 3) the ejection of low axial momentum fluid away from the wall.

This more complete description of wall turbulence is consistent with the principle of surface renewal[6] which will now be utilized in the development of a model for both the turbulent and laminar domains associated with boundary-layer flow. This modeling concept has been previously adapted to turbulent tube[7,8] and boundary-layer flow,[9-10] but has been restricted to Reynolds numbers well above transition.

To provide a framework for the analysis to follow, Fig. 1 illustrates the essential characteristics of the problem at hand. The usual boundary-layer thickness is indicated by δ, and x_t marks the transition from laminar to turbulent flow. Based on the flow visualization studies stated previously, the turbulent flow that exists in the wall region for $x > x_t$, at any given instant of time t, is assumed to consist of numerous packages of fluid which are undergoing unsteady molecular momentum transfer. Since these elements of fluid are identified with the bursting cycle at this instant t, each package has been in the wall region for a specific length of time [contact time] θ. This mosaic of elements can be characterized by an axially local mean burst period or residence time τ_x. In connection with the region $x < x_t$, this laminar flow condition can be described in the context of the more general turbulent flow condition with τ_x assumed to be infinite.

Analysis

In order to develop a theoretical model of turbulent boundary-layer flow which is consistent with the bursting mechanism, the following key considerations are involved: 1) an approximate analysis of the unsteady momentum transfer that occurs within any individual element of fluid in the wall region between inrush and ejection, 2) the evaluation of the spatial average or time average transport properties in terms of the bursting period τ_x, and 3) the formation of a relationship for τ_x in terms of the local mean wall shear stress σ_{ox}. Additional aspects of the analysis involve 4) the introduction of a reasonable assumption regarding the parameter U_i, and 5) the use of an approximation for the local Fanning friction factor.

Instantaneous Transport

With the inrush phase of an individual burst assumed to have just occurred, the momentum and continuity equations that approximately prescribe the unsteady transport process within the element of fluid that has been brought into the wall region take the forms

$$\partial u/\partial\theta + u\,\partial u/\partial x + v\,\partial u/\partial y = \nu\,\partial^2 u/\partial y^2 \tag{1}$$

$$\partial u/\partial x + \partial v/\partial y = 0 \tag{2}$$

for incompressible and uniform property conditions; θ is the instantaneous contact time. The accompanying initial and boundary conditions are taken as

$$u = U_i \quad \text{at} \quad \theta = 0 \tag{3}$$

$$u = 0 \quad \text{at} \quad y = 0 \tag{4}$$

$$u = U_i \quad \text{as} \quad y \to \infty \tag{5}$$

where U_i is the velocity at the first instant of renewal. The x-boundary condition will be given later.

Although detail flow visualization studies indicate that the inrush does not bring fluid into direct contact with the surface,

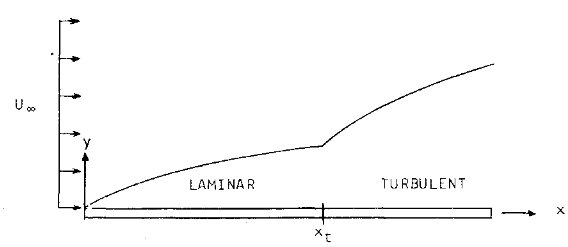

Fig. 1 Hydrodynamic boundary layer.

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the assumption embodied in Eq. (3) that fresh fluid is brought to the wall itself has been found to be a reasonable approximation for the analysis of momentum transfer.[7-12]

Because Eqs. (1) and (2) define the unsteady momentum transport process over a very brief period of time, it is now assumed that these equations can be approximated by

$$\partial u/\partial\theta + u\,\partial\bar{u}/\partial x + v\,\partial\bar{u}/\partial y = \nu\,\partial^2 u/\partial y^2 \qquad (6)$$

$$\partial\bar{u}/\partial x + \partial\bar{v}/\partial y = 0 \qquad (7)$$

where $\bar{u}$ and $\bar{v}$ represent the mean profiles. Although the terms $\partial u/\partial x$ and $\partial u/\partial y$ actually range from maximum absolute values at $\theta = 0$ to minimum values at the end of the residence time, it is assumed that the above substitution reasonably approximates the contribution of these terms. This point has recently been substantiated in the context of convective heat transfer for turbulent tube flow.[12]

Mean Transport

Because the bursting process is randomly distributed in time and space, the spatial average velocity profiles, $\bar{u}$ and $\bar{v}$, can be written in terms of the statistical contact time distribution $\phi(\theta)$ as

$$\bar{u} = \int_0^\infty u(\theta)\phi(\theta)\,d\theta \qquad (8)$$

$$\bar{v} = \int_0^\infty v(\theta)\phi(\theta)\,d\theta \qquad (9)$$

This distribution is defined such that the product $\phi(\theta)\,d\theta$ represents the fraction of elements with contact time between θ and $\theta + d\theta$. Interestingly, predictions for the mean transport properties associated with other turbulent flow conditions have been found to be insensitive to the form of this distribution.[8] Therefore, the convenient random distribution proposed by Danckwerts[6]

$$\phi(\theta) = (1/\tau_x)\exp(-\theta/\tau_x) \qquad (10)$$

will be utilized here, with the mean residence time taken as a function of x. Notice that $\bar{u}$ and $\bar{v}$ can be written in terms of the Laplace Transforms $L\{u(\theta)\}$ and $L\{v(\theta)\}$ as

$$\bar{u} = L\{u(\theta)\}/\tau_x \qquad (11)$$

$$\bar{v} = L\{v(\theta)\}/\tau_x \qquad (12)$$

With this understanding, the transformation of Eq. (6) into the mean time domain takes the form

$$(\bar{u} - U_i)/\tau_x + \bar{u}\,\partial\bar{u}/\partial x + \bar{v}\,\partial\bar{u}/\partial y = \nu\,\partial^2\bar{u}/\partial y^2 \qquad (13)$$

This expression together with the continuity equation which has already been written in the mean form, Eq. (7), is coupled with the appropriate boundary conditions in order to define the mean transport process. The y-conditions are readily written from Eqs. (4) and (5) as

$$\bar{u} = 0 \quad \text{at} \quad y = 0 \qquad (14)$$

$$\bar{u} = U_i \quad \text{as} \quad y \to \infty \qquad (15)$$

It is observed that Eqs. (7) and (13) appropriately describe the laminar flow situation for $\tau_x = \infty$ and $U_i = U_\infty$. Hence, the x-condition can now be established for the mean domain as

$$\bar{u} = U_\infty \quad \text{at} \quad x = 0 \qquad (16)$$

Although an exact solution of these equations is preferred, a preliminary solution can be readily obtained by use of the approximate integral solution technique. Integration and appropriate substitution leads to an integral momentum equation of the form

$$\int_0^l (\bar{u} - U_i)/\tau_x\,dy + \partial/\partial x \int_0^l \bar{u}(\bar{u} - U_i)\,dy = -\nu\,\partial\bar{u}/\partial y|_0 \qquad (17)$$

where l is defined as the depth of molecular penetration; l therefore describes the extent of the wall region. It will be shown that $l = \delta$ in the laminar domain, but that $l \ll \delta$ in the turbulent region.

With the y-boundary condition away from the wall rewritten as

$$\bar{u} = U_i \quad \text{at} \quad y = l \qquad (18)$$

$\bar{u}$ can be represented by

$$\bar{u}/U_i = \tfrac{3}{2}y/l - \tfrac{1}{2}(y/l)^3 \qquad (19)$$

The substitution of this assumed third order Kantorovich profile into Eq. (17) leads to the following differential equation for l:

$$\tfrac{13}{70}U_i\,dl^2/dx + l^2/\tau_x = 4\nu \qquad (20)$$

or

$$\tfrac{13}{70}U_i/U_\infty\,d\,Re_l^2/d\,Re_x + Re_l^2/(U_\infty^2\tau_x/\nu) = 4 \qquad (21)$$

Formulation for τ_x

An expression now can be written for the local mean wall shear stress of the form

$$\sigma_{ox} = \mu\,\partial\bar{u}/\partial y|_0 = \tfrac{3}{2}\mu\,U_i/l \qquad (22)$$

or

$$Re_l = \tfrac{3}{2}(2/f_x)U_i/U_\infty \qquad (23)$$

The substitution of this expression for l into Eqs. (20) or (21) provides a relationship for τ_x in terms of the local Fanning friction factor of the form

$$U^{*2}\tau_x/\nu = \tfrac{9}{4}(2/f_x)(U_i/U_\infty)[4 - (\tfrac{26}{70})(\tfrac{9}{4})(U_i/U_\infty)^3(2/f_x) \times d(2/f_x)/d\,Re_x]^{-1} \qquad (24)$$

Assumption for U_i

U_i has been previously found to be reasonably approximated by the bulk stream velocity U_b for tube flow.[7,11] Therefore, based on the usual boundary-layer substitution of 0.8 U_∞ for U_b, U_i will be set equal to 0.8 U_∞ in the present analysis. This assumption accounts for the fact that fluid elements move from various locations within the boundary-layer region $l < y < \delta$ to the vicinity of the wall.

Formulation for f_x and δ

Based on the usual approximate integral technique, $\bar{u}$ is assumed to take the following form away from the wall

$$\bar{u}/U_\infty = (y/\delta)^{1/7} \qquad (25)$$

With the local mean turbulent shear stress taken analogically from tube flow as

$$\sigma_{ox} = 0.0228\,\rho U_\infty^2\,Re_\delta^{-0.25} \qquad (26)$$

where U_b has been replaced by 0.8 U_∞, an expression can be written for the turbulent boundary-layer thickness of the form[13,14]

$$\tfrac{7}{72}\,d\delta/dx = \sigma_{ox}/(\rho U_\infty^2) \qquad (27)$$

The substitution of Eq. (26) into this expression gives

$$\delta^{0.25}\,d\delta = 0.0228(\nu/U_\infty)^{0.25}\tfrac{72}{7}\,dx \qquad (28)$$

The solution of this equation with δ set equal to zero gives

$$Re_\delta = 0.376\,Re_x^{0.8} \qquad (29)$$

and

$$f_x/2 = 0.0295\,Re_x^{-0.2} \qquad (30)$$

This expression for f_x has been found to be in good agreement with experimental data for turbulent boundary-layer flow with no axial pressure gradient for $Re_x < 10^7$. For $Re_x > 10^7$, other relationships should be used for f_x such as[14]

$$f_x = [2\log Re_x - 0.65]^{-2.3} \qquad (31)$$

With the effect of the laminar boundary layer accounted for by use of the more representative boundary condition $\delta = \delta_t$ at $x = x_t$ and assuming, as a first approximation, that beyond the point of transition the turbulent boundary layer behaves as if it were turbulent from the leading edge, Eqs. (27) and (30) lead to a more representative expression for δ in the region $Re_x > Re_{x_t}$ of the form

$$Re_\delta = Re_{\delta_t} + 0.379\,(Re_x^{0.8} - Re_{x_t}^{0.8}) \qquad (32)$$

Since l in the present analysis is equal to δ for $Re_x < Re_{x_t}$, an expression can be obtained for Re_δ associated with laminar flow by the solution of Eqs. (20) and (23) with τ_x set equal to infinity and with U_i equal to U_∞. The resulting approximate expression for Re_δ or Re_l is

$$Re_\delta = Re_l = 4.64\, Re_x^{0.5} \tag{33}$$

for $Re_x < Re_{x_t}$. Hence, Re_{δ_t} can be replaced by $4.64\, Re_{x_t}\, 0.5$.

Results and Discussion

Predictions for l

Predictions for Re_l given by Eq. (23) are shown by Fig. 2, with f_x given by Eq. (30) and with U_i set equal to $0.8\ U_\infty$. Figure 2 also shows the calculations for δ in terms of Re_δ vs Re_x given by Eqs. (29) and (33). The paths followed by δ for transition Reynolds numbers of 3×10^4, 8×10^4, 5×10^5, and 10^6 are illustrated. The extent of the wall region as represented by l is equal to δ for laminar flow and lies well inside δ for turbulent flow conditions.

Predictions for τ_x

With f_x given by Eq. (30) for $Re_x > Re_{x_t}$, Eq. (24) leads to expressions for τ_x of the forms

$$U_\infty^2 \tau_x/\nu \equiv \tau_x^+ \doteq 2590(U_i/U_\infty)^2 Re_x^{0.4} \times [4 - 1920(U_i/U_\infty)^3 Re_x^{-0.6}]^{-1} \tag{34}$$

and

$$U^{*2} \tau_x/\nu \equiv \tau_x^* = 76.5(U_i/U_\infty)^2 Re_x \times [4 - 1920(U_i/U_\infty)^3 Re_x^{-0.6}]^{-1} \tag{35}$$

Predictions for τ_x are shown in Fig. 3 in terms of the dimensionless frequency $1/\tau_x^+$ vs Re_x for U_i equal to $0.8\ U_\infty$. A most interesting feature of these results is the fact that $\tau_x \to \infty$ in the vicinity of $Re_x = 1.5 \times 10^4$. Below this Re_x the proposed analysis indicates that turbulent flow cannot exist. Although this analysis specifies an absolute minimum value of Re_x at which turbulence can exist, it does not preclude the existence of a higher transitional Reynolds number. In this connection, for $Re_x < Re_{x_t}$, the substitution of the friction fact or for laminar flow into Eq. (24) appropriately give $\tau_x = \infty$.

Another point of interest is that for a given Re_{x_t}, the value of maximum turbulence activity occurs at Re_{x_t} or near 8×10^4, whichever is greater. Hence, for $Re_{x_t} < 8 \times 10^4$, $1/\tau_x^+$ steps abruptly from zero for $Re_x < Re_{x_t}$ to a finite value at Re_{x_t}, and then increases to a maximum value at $Re_x \simeq 8 \times 10^4$, after which $1/\tau_x^+$ decreases. However, for $Re_{x_t} > 8 \times 10^4$, $1/\tau_x^+$ rises from zero to a maximum value at Re_{x_t}, after which $1/\tau_x^+$ decreases with increasing Re_x in the same fashion as above.

The effect of the convective terms on the predictions for τ_x can be obtained by consideration of the limit of Eqs. (34) or (35) as Re_x becomes large. For this limiting condition, Eq. (34) reduces to

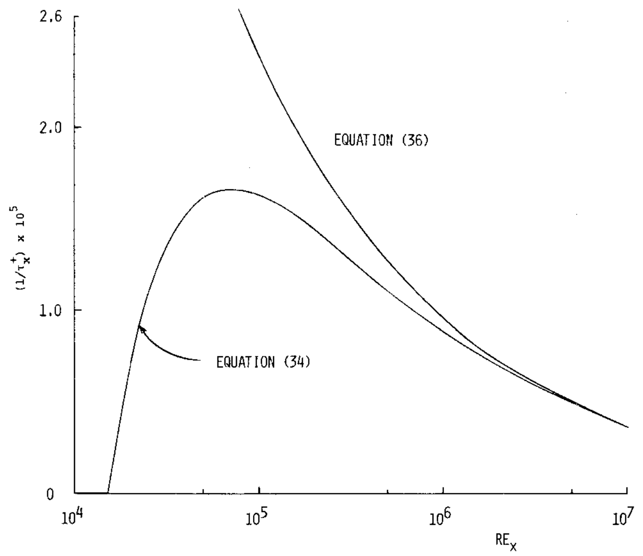

Fig. 3 Predictions for τ_x.

$$\tau_x^+ = 2590(U_i/U_\infty)^2 Re_x^{-0.4} = \tfrac{9}{16}(2/f_x)^2(U_i/U_\infty)^2 \tag{36}$$

This expression can also be obtained directly by the solution of Eq. (20) with the convective term dl^2/dx assumed to be negligibly small. Equations (34) and (36) are seen to come together for $Re_x \gtrsim 10^7$. Figure 3 suggests that the convective influence can be categorized as 1) negligible for $Re_x \gtrsim 10^7$, 2) secondary for $10^7 \gtrsim Re_x \gtrsim 10^5$, 3) strong for $10^5 > Re_x > Re_{x_t}$, and 4) totally dominant for $Re_x < Re_{x_t}$. Because of the dramatic stabilizing effect of the convective terms on boundary-layer flow as reflected in the present analysis, it is perhaps more than mere coincidence that Re_{x_t} generally occurs in the vicinity of $Re_x \simeq 10^5$.

Consideration is now turned to a comparison of experimental measurements for τ_x with predictions based on Eq. (35). However, it should first be noted that several techniques have been employed in measurements of the bursting period which result in noticeable variations in the data. The two most popular methods involve 1) visual observation,[3,4] and 2) anemometry.[15-17] The visual detection of bursts have all been based on observations at a plane away from the wall. The anemometry techniques have provided measurements both at and away from the wall. Because the present analysis is based on the simplified assumption that turbulence extends to the wall itself, attention will be focused upon measurements obtained by flush mounted anemometer probes. Measurements for τ_x obtained in this manner simulate the apparent bursting period seen by the wall.

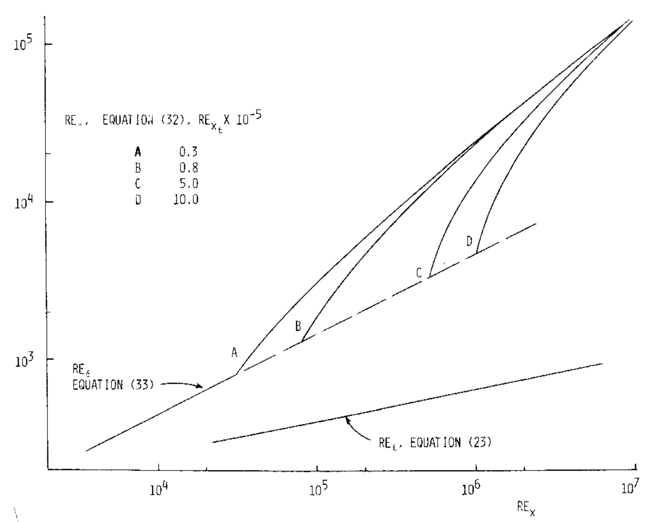

Fig. 2 Predictions for RE_l.

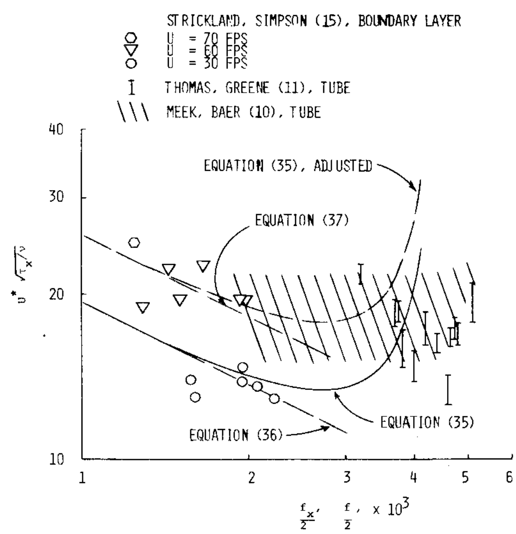

Fig. 4 Comparison of predictions for τ_x with experimental data.

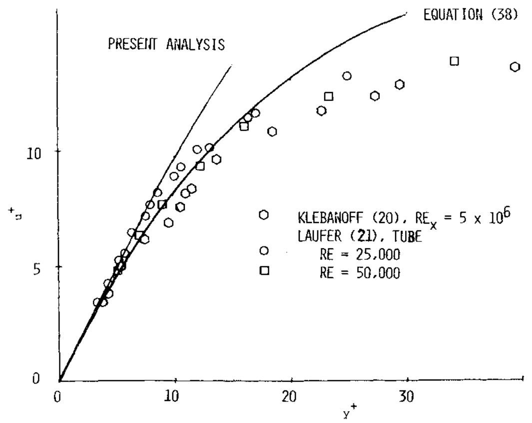

Fig. 5 Comparison of predictions for u with experimental data.

Experimental measurements for τ_x obtained by the use of flush mounted anemometer probes are shown in Fig. 4 for both boundary-layer[15] and tube flow[16,17] in terms of $(\tau^*)^{1/2}$ vs $f_x/2$ or $f/2$. Although the boundary-layer data possess a marked inconsistency between the measurement made for $U_\infty = 70$ and 60 fps vs 30 fps, the data taken at the higher value of U_∞ are seen to be fairly consistent with the tube flow data. Predictions for τ_x based on Eq. (35) are shown along with the exact solution for large Re_x given by[10]

$$U^*(\tau_x/\nu)^{1/2} = (\tau^*)^{1/2} = (2/f_x)^{1/2} U_i/U_\infty \tag{37}$$

With Eq. (35) raised 25% in order to adjust for the error introduced by the approximate integral technique, the predictions are seen to be in exceptional agreement with the experimental data for $U_i = 70$ and 60 fps.

Whereas the convective terms are responsible for stabilizing boundary-layer flow, for fully developed tube or channel flow the axial pressure gradient is the stabilizing factor.[12] Although the boundary-layer data essentially lie in the region in which the stabilizing convective influence is secondary, the most recent tube flow data[17] appear to reflect the stabilizing effect of the axial pressure gradient.

Prediction for $\bar{u}$

Equation (19), with l given by Eq. (23), is compared with experimental data for large Re_x in Fig. 5. The error inherent in the approximate integral technique utilized in this analysis is reflected in the obvious difference between the predictions and data for $y^+ > 10$. This error in $\bar{u}$ can be seen by a consideration of the exact solution for negligible convective effects, i.e.,[18]

$$\bar{u}/U_i = 1 - \exp\left[y/(\nu\tau_x)^{1/2}\right] \tag{38}$$

With τ_x given by Eq. (37), this expression is seen to follow the data trend throughout most of the wall region. Hence, it appears that viable predictions for $\bar{u}$ within the entire wall region for values of Re_x within the transition region will require a more accurate solution of the modeling equations. Of course, for laminar conditions $\tau_x \to \infty$ and the present analysis reduces to the familiar Pohlhausen[19] solution.

Summary

The key contribution of this paper is the inclusion of the effects of axial convection in the context of the basic surface renewal modeling approach. This model provides a foundation for the unified analysis of both turbulent and laminar boundary-layer flow. Although the approximate nature of the analysis introduces some error, the proposed model actually predicts that stability is brought about by the influence of the convective terms in the vicinity of the known transition region. The modeling parameter involved in this analysis, τ_x, has physical significance in that it is a measure of the bursting period reported in flow visualization studies. It is hoped that the present preliminary surface renewal based formulation for boundary-layer flow will provide direction and incentive for further development of this powerful modeling concept.

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